On quantum Cayley graphs

Mateusz Wasilewski

IM PAN

16 Oct 2023

Outline

- Infinite discrete quantum groups can be non-unimodular

Outline

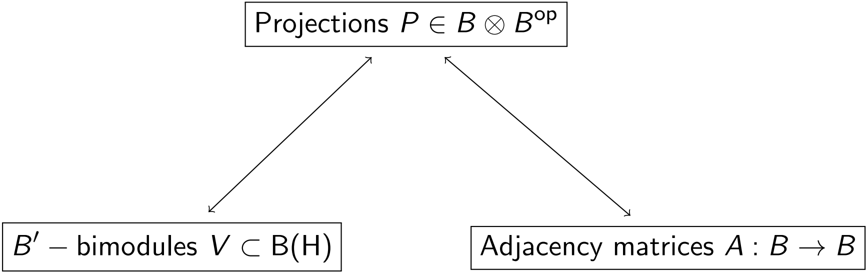
- Infinite discrete quantum groups can be non-unimodular
- We need quantum graphs that are both infinite and non-tracial

Outline

- Infinite discrete quantum groups can be non-unimodular
- We need quantum graphs that are both infinite and non-tracial
- Quantum Cayley graphs: covariant quantum adjacency matrices

Three definitions

(B, ψ) - fin. dim. C^*-algebra with $mm^* = \mathrm{Id}$



Weaver/Duan-Severini-Winter

Quantum relations

Let $B \subset \mathrm{B(H)}$. Quantum relation - a weak* closed B'-bimodule $V \subset \mathrm{B(H)}$.

Weaver/Duan-Severini-Winter

Quantum relations

Let $B \subset \mathrm{B(H)}$. Quantum relation - a weak* closed B'-bimodule $V \subset \mathrm{B(H)}$.

Independent of the representation!

Weaver/Duan-Severini-Winter

Quantum relations

Let $B \subset \mathrm{B(H)}$. Quantum relation - a weak* closed B'-bimodule $V \subset \mathrm{B(H)}$.

Independent of the representation!

From projections to relations

Any $P \in B \otimes B^{\mathrm{op}}$ acts on $\mathrm{B(H)}$ via left-right multiplication. The image will be a B'-bimodule V.

From bimodules to projections

Inverse of a weight

Any faithful functional $\psi : B \to \mathbb{C}$ gives rise to an operator valued weight $\psi^{-1} : \mathrm{B(H)} \to B'$.

From bimodules to projections

Inverse of a weight

Any faithful functional $\psi : B \to \mathbb{C}$ gives rise to an operator valued weight $\psi^{-1} : \mathrm{B(H)} \to B'$. Equality $mm^* = \mathrm{Id}$ ensures that ψ^{-1} is a conditional expectation.

From bimodules to projections

Inverse of a weight

Any faithful functional $\psi : B \to \mathbb{C}$ gives rise to an operator valued weight $\psi^{-1} : \mathrm{B(H)} \to B'$. Equality $mm^* = \mathrm{Id}$ ensures that ψ^{-1} is a conditional expectation.

Hilbert bimodules

Using ψ^{-1} we equip any B'-bimodule V with a structure of a Hilbert B'-module.

From bimodules to projections

Inverse of a weight

Any faithful functional $\psi : B \to \mathbb{C}$ gives rise to an operator valued weight $\psi^{-1} : \mathrm{B(H)} \to B'$. Equality $mm^* = \mathrm{Id}$ ensures that ψ^{-1} is a conditional expectation.

Hilbert bimodules

Using ψ^{-1} we equip any B'-bimodule V with a structure of a Hilbert B'-module.

Bimodular projections

The orthogonal projection $\widetilde{P} : \mathrm{B(H)} \to V$ is a B'-bimodular map, hence it is represented by an element $P \in B \otimes B^{\mathrm{op}}$.

Generalized Choi matrix

KMS inner product

Endow B with the inner product $\langle a, b \rangle := \psi(a^* \sigma_{-\frac{i}{2}}(b))$.

Generalized Choi matrix

KMS inner product

Endow B with the inner product $\langle a, b\rangle := \psi(a^*\sigma_{-\frac{i}{2}}(b))$.

Generalized Choi matrix

There is a one-to-one map $\mathrm{Choi} : \mathrm{End}(B) \to B \otimes B^{\mathrm{op}}$ given by $|a\rangle\langle b| \mapsto a \otimes b^*$.

Generalized Choi matrix

KMS inner product

Endow B with the inner product $\langle a, b\rangle := \psi(a^* \sigma_{-\frac{i}{2}}(b))$.

Generalized Choi matrix

There is a one-to-one map $\mathrm{Choi} : \mathrm{End}(B) \to B \otimes B^{\mathrm{op}}$ given by $|a\rangle\langle b| \mapsto a \otimes b^*$.

- $\mathrm{Choi}(S)\,\mathrm{Choi}(T) = \mathrm{Choi}(m(S \otimes T)m^*)$

Generalized Choi matrix

KMS inner product

Endow B with the inner product $\langle a, b\rangle := \psi(a^*\sigma_{-\frac{i}{2}}(b))$.

Generalized Choi matrix

There is a one-to-one map $\mathrm{Choi}: \mathrm{End}(B) \to B \otimes B^{\mathrm{op}}$ given by $|a\rangle\langle b| \mapsto a \otimes b^*$.

- $\mathrm{Choi}(S)\,\mathrm{Choi}(T) = \mathrm{Choi}(m(S \otimes T)m^*)$
- $\mathrm{Choi}(T) = (\mathrm{Choi}(T))^*$ iff T is $*$-preserving

Generalized Choi matrix

KMS inner product

Endow B with the inner product $\langle a, b\rangle := \psi(a^*\sigma_{-\frac{i}{2}}(b))$.

Generalized Choi matrix

There is a one-to-one map $\mathrm{Choi} : \mathrm{End}(B) \to B \otimes B^{\mathrm{op}}$ given by $|a\rangle\langle b| \mapsto a \otimes b^*$.

- $\mathrm{Choi}(S)\,\mathrm{Choi}(T) = \mathrm{Choi}(m(S \otimes T)m^*)$
- $\mathrm{Choi}(T) = (\mathrm{Choi}(T))^*$ iff T is $*$-preserving
- Choi(T) is positive iff T is completely positive

Generalized Choi matrix

KMS inner product

Endow B with the inner product $\langle a, b\rangle := \psi(a^*\sigma_{-\frac{i}{2}}(b))$.

Generalized Choi matrix

There is a one-to-one map $\mathrm{Choi} : \mathrm{End}(B) \to B \otimes B^{\mathrm{op}}$ given by $|a\rangle\langle b| \mapsto a \otimes b^*$.

- $\mathrm{Choi}(S)\,\mathrm{Choi}(T) = \mathrm{Choi}(m(S \otimes T)m^*)$
- $\mathrm{Choi}(T) = (\mathrm{Choi}(T))^*$ iff T is $*$-preserving
- Choi(T) is positive iff T is completely positive
- $\mathrm{Choi}(T)$ is invariant under the flip iff T is KMS symmetric

Quantum adjacency matrices

Schur idempotents

A completely positive map $A : B \to B$ is a quantum adjacency matrix if $m(A \otimes A)m^* = A$.

Quantum adjacency matrices

Schur idempotents

A completely positive map $A : B \to B$ is a quantum adjacency matrix if $m(A \otimes A)m^* = A$.

Undirected quantum graphs

A quantum graph is called undirected if its quantum adjacency matrix is KMS symmetric.

Quantum adjacency matrices

Schur idempotents

A completely positive map $A : B \to B$ is a quantum adjacency matrix if $m(A \otimes A)m^* = A$.

Undirected quantum graphs

A quantum graph is called undirected if its quantum adjacency matrix is KMS symmetric.

Degree matrix

The matrix $D := A\mathbb{1}$ is called the degree matrix of the quantum graph.

What changes in infinite dimension?

Now B is an infinite direct sum of matrix algebras

What changes in infinite dimension?

Now B is an infinite direct sum of matrix algebras

Bounded degree

Normal cp maps $A : B \to B$ satisfying $m(A \otimes A)m^* = A$ define a class of quantum graphs of bounded degree.

What changes in infinite dimension?

Now B is an infinite direct sum of matrix algebras

Bounded degree

Normal cp maps $A : B \to B$ satisfying $m(A \otimes A)m^* = A$ define a class of quantum graphs of bounded degree.

Projections

The projections $P \in B \overline{\otimes} B^{\text{op}}$ of bounded degree are exactly the ones with $(\text{Id} \otimes \psi^{\text{op}})(P) \in B$.

What changes in infinite dimension?

Now B is an infinite direct sum of matrix algebras

Bounded degree

Normal cp maps $A : B \to B$ satisfying $m(A \otimes A)m^* = A$ define a class of quantum graphs of bounded degree.

Projections

The projections $P \in B \overline{\otimes} B^{\mathrm{op}}$ of bounded degree are exactly the ones with $(\mathrm{Id} \otimes \psi^{\mathrm{op}})(P) \in B$.

Bimodules

Weak* bimodules $V \subset \mathrm{B(H)}$ of bounded degree are exactly the self-dual ones (for all representations $B \subset \mathrm{B(H)}$).

Compact/discrete quantum groups

Compact quantum groups

C^*-algebra $C(\mathbb{G})$ with a coassociative $*$-homomorphism $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ + density conditions.

Compact/discrete quantum groups

Compact quantum groups

C^*-algebra $C(\mathbb{G})$ with a coassociative $*$-homomorphism $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ + density conditions.

- Haar measure h exists

Compact/discrete quantum groups

Compact quantum groups

C^*-algebra $C(\mathbb{G})$ with a coassociative $*$-homomorphism $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ + density conditions.

- Haar measure h exists
- Good representation theory (Peter-Weyl etc.)

Compact/discrete quantum groups

Compact quantum groups

C^*-algebra $C(\mathbb{G})$ with a coassociative $*$-homomorphism $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ + density conditions.

- Haar measure h exists
- Good representation theory (Peter-Weyl etc.)

Discrete dual

Set $\ell^\infty(\Gamma) := \ell^\infty - \bigoplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} M_{d_\alpha}$. One can define $\widehat{\Delta} : \ell^\infty(\Gamma) \to \ell^\infty(\Gamma) \overline{\otimes} \ell^\infty(\Gamma)$.

Compact/discrete quantum groups

Compact quantum groups

C^*-algebra $C(\mathbb{G})$ with a coassociative $*$-homomorphism $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ + density conditions.

- Haar measure h exists
- Good representation theory (Peter-Weyl etc.)

Discrete dual

Set $\ell^\infty(\Gamma) := \ell^\infty - \bigoplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} M_{d_\alpha}$. One can define $\widehat{\Delta} : \ell^\infty(\Gamma) \to \ell^\infty(\Gamma) \overline{\otimes} \ell^\infty(\Gamma)$.
There are also explicit formulas for the left and right Haar measures.

Covariant adjacency matrices

Convolution operators

If $A_1 x := P_1 * x$ and $A_2 := P_2 * x$ then $A := m(A_1 \otimes A_2)m^*$ is given by $Ax = P_1 P_2 * x$.

Covariant adjacency matrices

Convolution operators

If $A_1 x := P_1 * x$ and $A_2 := P_2 * x$ then $A := m(A_1 \otimes A_2)m^*$ is given by $Ax = P_1 P_2 * x$.

Covariant adjacency matrices

Convolution operator $Ax := P * x$ is a quantum adjacency matrix iff P is a projection.

Covariant adjacency matrices

Convolution operators

If $A_1 x := P_1 * x$ and $A_2 := P_2 * x$ then $A := m(A_1 \otimes A_2)m^*$ is given by $Ax = P_1 P_2 * x$.

Covariant adjacency matrices

Convolution operator $Ax := P * x$ is a quantum adjacency matrix iff P is a projection.

- KMS symmetric iff P is invariant under the unitary antipode R

Covariant adjacency matrices

Convolution operators

If $A_1 x := P_1 * x$ and $A_2 := P_2 * x$ then $A := m(A_1 \otimes A_2)m^*$ is given by $Ax = P_1 P_2 * x$.

Covariant adjacency matrices

Convolution operator $Ax := P * x$ is a quantum adjacency matrix iff P is a projection.

- KMS symmetric iff P is invariant under the unitary antipode R
- GNS symmetric iff P is invariant under the antipode S.

Quantum Cayley graphs

Generating projections

A projection $P \in \ell^{\infty}(\mathbb{\Gamma})$ is generating if $\bigvee_{n\geqslant 1}[P^{*n}] = \mathbb{1}$, where $[T]$ is the support projection of T.

Quantum Cayley graphs

Generating projections

A projection $P \in \ell^{\infty}(\mathbb{\Gamma})$ is generating if $\bigvee_{n \geqslant 1}[P^{*n}] = \mathbb{1}$, where $[T]$ is the support projection of T.

Quantum Cayley graph

Let $P \in \ell^{\infty}(\mathbb{\Gamma})$ be a generating projection such that $R(P) = P$ and $\varepsilon(P) = 0$. We call the pair $(\ell^{\infty}(\mathbb{\Gamma}), P * \cdot)$ a quantum Cayley graph of $\mathbb{\Gamma}$.

Quantum Cayley graphs

Generating projections

A projection $P \in \ell^\infty(\mathbb{\Gamma})$ is generating if $\bigvee_{n \geqslant 1} [P^{*n}] = \mathbb{1}$, where $[T]$ is the support projection of T.

Quantum Cayley graph

Let $P \in \ell^\infty(\mathbb{\Gamma})$ be a generating projection such that $R(P) = P$ and $\varepsilon(P) = 0$. We call the pair $(\ell^\infty(\mathbb{\Gamma}), P * \cdot)$ a quantum Cayley graph of $\mathbb{\Gamma}$.

Bi-Lipschitz equivalence

For two quantum Cayley graphs of $\mathbb{\Gamma}$ the corresponding quantum metric spaces are bi-Lipschitz equivalent.

The bimodule picture

Decomposition of bimodules

$B := \ell^\infty(\mathbb{\Gamma}) \subset \mathrm{B}(\mathrm{H})$, where $\mathrm{H} := \bigoplus_\alpha \mathbb{C}^{d_\alpha}$. Then a weak* closed B'-bimodule $V \subset \mathrm{B}(\mathrm{H})$ is just a collection of subspaces $V_{\alpha\beta} \subset \mathrm{B}(\mathbb{C}^{d_\alpha}, \mathbb{C}^{d_\beta})$.

The bimodule picture

Decomposition of bimodules

$B := \ell^\infty(\mathbb{\Gamma}) \subset \mathrm{B}(\mathsf{H})$, where $\mathsf{H} := \bigoplus_\alpha \mathbb{C}^{d_\alpha}$. Then a weak* closed B'-bimodule $V \subset \mathrm{B}(\mathsf{H})$ is just a collection of subspaces $V_{\alpha\beta} \subset \mathrm{B}(\mathbb{C}^{d_\alpha}, \mathbb{C}^{d_\beta})$.

The central case

Suppose $P = \mathbb{1}_\gamma$, where γ is a generating representation of $\mathbb{G}$. Then $V_{\alpha\beta} = \mathrm{span}\{T(\mathrm{Id} \otimes |v\rangle) : v \in \mathsf{H}_\gamma, T \in \mathrm{Mor}(\alpha \otimes \gamma, \beta)\}$, where $\mathrm{Id} \otimes |v\rangle : \mathsf{H}_\alpha \to \mathsf{H}_\alpha \otimes \mathsf{H}_\gamma$ is given by $(\mathrm{Id} \otimes |v\rangle)(w) := w \otimes v$.

Examples

Free unitary quantum groups

$\mathbb{\Gamma} = \widehat{U_F^+}$ and $P = \mathbb{1}_u + \mathbb{1}_{\overline{u}}$. Because $u^{\otimes n}$ is irreducible the directed quantum graph $\mathbb{1}_u * \cdot$ is acyclic. Possibly we get a tree.

Examples

Free unitary quantum groups

$\mathbb{\Gamma} = \widehat{U_F^+}$ and $P = \mathbb{1}_u + \mathbb{1}_{\overline{u}}$. Because $u^{\otimes n}$ is irreducible the directed quantum graph $\mathbb{1}_u * \cdot$ is acyclic. Possibly we get a tree.

$\widehat{SU(2)}$

This is related to well-studied quantum random walks on the dual of $SU(2)$ (Biane).

Frucht's theorem

Theorem (Frucht)

Every finite group is an automorphism group of a finite graph.

Frucht's theorem

Theorem (Frucht)

Every finite group is an automorphism group of a finite graph.

Theorem (Watkins' conjecture)

For a finite group G there exists a Cayley graph with all automorphisms coming from G unless:

- *G is abelian with elements of order 2;*
- *G is generalized dicyclic;*
- *G belongs to the finite set of exceptional examples.*

Quantum Frucht?

Theorem (Banica-McCarthy)

Some finite quantum groups, e.g. $\widehat{S_3}$, do not arise as quantum automorphism groups of graphs.

Quantum Frucht?

Theorem (Banica-McCarthy)

Some finite quantum groups, e.g. $\widehat{S_3}$, do not arise as quantum automorphism groups of graphs.

Quantum graphs enter the stage

Is it possible that every finite quantum group is a quantum automorphism group of a finite **quantum** graph?

Quantum Frucht?

Theorem (Banica-McCarthy)

Some finite quantum groups, e.g. $\widehat{S_3}$, do not arise as quantum automorphism groups of graphs.

Quantum graphs enter the stage

Is it possible that every finite quantum group is a quantum automorphism group of a finite **quantum** graph? I don't know yet, but there are some partial results.

Duals of finite groups

Separating matrix coefficients

Let $\Gamma := \widehat{G}$. P – rank one projection in $\mathbb{C}[G]$. If the Fourier transform of P separates points of G then the quantum automorphism group of $(\mathbb{C}[G], P * \cdot)$ is equal to Γ.

Duals of finite groups

Separating matrix coefficients

Let $\Gamma := \widehat{G}$. P – rank one projection in $\mathbb{C}[G]$. If the Fourier transform of P separates points of G then the quantum automorphism group of $(\mathbb{C}[G], P * \cdot)$ is equal to Γ.

Examples

If G admits a faithful irreducible representation then a generic choice works. Examples include permutation groups, dihedral groups etc.

Thank you for your attention!